gravity-turn solution from low-velocity terminal descent to complete descent from a low parking orbit. Although the solution to the modified gravity-turn problem can be obtained exactly as a special function, a first-order correction to the classical gravity-turn solution can extend its validity to a larger range of descent velocities. The availability of the descent vehicle velocity as a function of the velocity pitch angle could, in principle, be used to reduce the computational burden on real-time guidance algorithms for future lander missions.

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Riccati Dichotomic Basis Method for Solving Hypersensitive Optimal Control Problems

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Introduction

MANY optimal control problems and their associated Hamiltonian boundary-value problems (HBVPs) that arise from the first-order optimality conditions are hypersensitive.[1–3] An optimal control problem is hypersensitive if the time interval of interest is long relative to the rate of expansion and contraction of the Hamiltonian dynamics in certain directions in a neighborhood of the optimal solution. Hypersensitive HBVPs are a challenge to solve numerically because they suffer from ill-conditioning as a result of extreme sensitivity to unknown boundary conditions. When the rates are fast in all directions, the HBVP and the optimal control problem are called completely hypersensitive; when the rates are fast only in certain some directions, the HBVP and the optimal control problem are called partially hypersensitive. In this Note we are interested in completely hypersensitive HBVPs.

The solution to a completely hypersensitive HBVP can be approximated by concatenating an initial boundary-layer segment, an equilibrium segment, and a terminal boundary-layer segment.[1–3] The initial boundary-layer segment has no unstable component in forward time, whereas the terminal boundary-layer segment has no unstable component in backward time. This three-segment approximation improves as the time interval of interest increases.

Recently, a new approach has been developed to solving completely hypersensitive nonlinear HBVPs arising in optimal control.[1–3] This method is inspired by the computational singular perturbation methodology for stiff initial-value problems.[4,5] The method uses a dichotomic basis to decompose the nonlinear Hamiltonian vector field into its contracting and expanding components, thus allowing the missing conditions required to specify the initial and terminal boundary-layer segments to be determined from partial equilibrium conditions. The key feature of the method is that, by using a dichotomic basis, the unstable (expanding) component of the Hamiltonian vector field can be eliminated, thereby removing the hypersensitivity. The solution of the initial boundary-layer segment is then found by integrating the stable component of the Hamiltonian vector field forward in time. Similarly, the solution of the terminal boundary-layer segment is found by integrating the unstable component of the Hamiltonian vector field backward in time.

In previous work on hypersensitive optimal control problems, the properties of a dichotomic basis were described, but no such basis had actually been found. Consequently, it was necessary to determine a solution to a completely hypersensitive HBVP using an approximate dichotomic basis. Although this method has shown some success, it can potentially fail because a sufficiently good approximate dichotomic basis can be difficult to determine. The major advancements of this research over previous work in the area of completely hypersensitive optimal control[1–3] are as follows: 1) the derivation of a dichotomic basis along the solution of a completely hypersensitive HBVP in the initial boundary layer and 2) the development of a successive approximation procedure to compute this dichotomic basis and the initial boundary-layer solution. The dichotomic basis described in this Note is found by solving a Riccati differential equation. The need for a successive approximation procedure arises because the solution in the initial boundary layer is not known a priori. The successive approximation procedure is illustrated on a problem in supersonic aircraft flight, and its range of applicability is assessed.

Hamiltonian Boundary-Value Problem

In this Note we are interested in the following class of optimal control problems. Find the piecewise continuous control $u(t) \in \mathbb{R}^m$ on $t \in [0, t_f]$ that minimizes the scalar cost functional

$$J = \int_0^{t_f} \mathcal{L}[x, u]\, \mathrm{d}t \tag{1}$$

subject to the differential constraint

$$\dot{x} = f(x, u) \tag{2}$$

and boundary conditions

$$x(0) = x_0, \qquad x(t_f) = x_f \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state.

The first-order necessary conditions for optimality lead to a HBVP for the extremal trajectories. The HBVP is composed of the Hamiltonian differential equations

$$\dot{x} = \left[\frac{\partial H^*}{\partial \lambda}\right]^T \equiv \left[H^*_\lambda\right]^T, \qquad \dot{\lambda} = -\left[\frac{\partial H^*}{\partial \lambda}\right]^T \equiv -\left[H^*_x\right]^T \tag{4}$$

where $\lambda(t) \in \mathbb{R}^n$ is the adjoint and $H^*(x, \lambda) = \mathcal{L}[x, u^*(x, \lambda)] + \lambda^T f[x, u^*(x, \lambda)]$ is the Hamiltonian evaluated at the optimal control $u^*(x, \lambda) = \arg\min_u H(x, \lambda, u)$. Points $p = (x, \lambda)$ lie in the $2n$-dimensional Hamiltonian phase space or, more simply, the phase space. Because J and $f(x, u)$ do not depend explicitly on time, H^* is constant along trajectories of Eq. (4). We use $\dot{p} = G(p)$ as an alternate expression for the Hamiltonian system in Eq. (4) and refer to $G(p)$ as the Hamiltonian vector field, where $G(p)$ is assumed to be continuously differentiable.

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Completely Hypersensitive HBVP

Boundary-Layer Structure

For sufficiently large values of t_f, the HBVP of Eq. (4) becomes completely hypersensitive. A detailed description of the structure of a completely hypersensitive HBVP has been described in Refs. 1 and 2 and is not repeated here. The key feature of a completely hypersensitive HBVP is that the optimal solution can be accurately approximated by the following three-segment trajectory:

$$\hat{p}(t) = \begin{cases} p_s(t) & 0 \le t \le t_{\text{ibl}} \\ \bar{p} & t_{\text{ibl}} \le t \le t_f - t_{\text{fbl}} \\ p_u(t) & t_f - t_{\text{fbl}} \le t \le t_f \end{cases} \tag{5}$$

where t_{ibl} and t_{fbl} denote the durations of the initial and terminal boundary-layer segments, respectively, and $\bar{p}$ is a saddle point[6] in the phase space. The segment defined by $p_s(t)$ is the solution of Eq. (4) with initial condition $p_s(0) = (x_0, \lambda_0)$ and λ_0 is chosen so that $p_s(0)$ lies in the stable manifold of $\bar{p}$ (Ref. 6). Similarly, the segment defined by $p_u(t)$ is the solution of Eq. (4) with terminal condition $p_u(t_f) = (x_f, \lambda_f)$, where $p_u(t)$ lies in the unstable manifold of $\bar{p}$. Consequently, the composite approximation is constructed by concatenating an initial boundary-layer segment on the stable manifold, an equilibrium segment, and a terminal boundary-layer segment that lies in the unstable manifold. The durations of the initial and terminal boundary-layer segments t_{ibl} and $t_f - t_{\text{fbl}}$ must be long enough to allow the boundary-layer segments to reach the equilibrium to sufficient accuracy in forward and backward time, respectively. Let ϵ denote the accuracy tolerance, and let $p^*(t)$ be the optimal trajectory of Eq. (4). Define the approximation error e by

$$e = \sup_{t \in [0, t_f]} \| \hat{p}(t) - p^*(t) \|_2 \tag{6}$$

If $e < \epsilon$, then the trajectory $\hat{p}(t)$ is a candidate approximation to $p^*(t)$.

Solution Structure in Boundary Layers

Two pieces of information can be deduced from the three-segment approximation of Eq. (5). First, the structure of the approximate solution in the initial and terminal boundary layer is the same; only the direction of time is different. Second, because the middle segment of the approximate solution is constant, it can be solved for algebraically. From these two observations it can be seen that in order to find a solution that satisfies the three-segment approximation of Eq. (5) it is sufficient to develop a method to find a solution in either the initial or terminal boundary layer. For simplicity in the remaining discussion, we focus our attention on developing a method to find a solution in the initial boundary layer.

Dichotomic Basis Method

Dichotomic Basis

Restricting attention to completely hypersensitive HBVPs, at each point in the $2n$-dimensional phase space along the optimal trajectory p^* there is an n-dimensional contracting subspace and an n-dimensional expanding subspace. Consequently, a neighboring optimal trajectory that begins in the contracting subspace will approach p^* in forward time while a neighboring optimal trajectory that begins in the expanding subspace will approach p^* in backward time. Assume that this property holds in a neighborhood $N \subset \mathbb{R}^{2n}$ of p^* and that the approximate three-segment trajectory of Eq. (5) lies in N. Let the columns of $D(p) \in \mathbb{R}^{2n \times 2n}$ form a continuously differentiable basis for $\mathbb{R}^{2n}$. Then $G(p)$ can be written in terms of $D(p)$ as

$$G(p) = D(p)v \tag{7}$$

where $v \in \mathbb{R}^{2n}$ are the components of $G(p)$ in the basis $D(p)$. For now let $p(t)$ be any trajectory that satisfies the differential equation

$$\dot{p} = G(p) \tag{8}$$

Differentiating along $p(t)$, we have

$$\dot{v} = (D^{-1}\mathcal{J}D - D^{-1}\dot{D})v \equiv \Lambda v \tag{9}$$

where $\mathcal{J} \equiv \partial G/\partial p$ is the Jacobian of $G(p)$. It can be seen that, along a trajectory $p(t)$ of Eq. (8), Eq. (9) is a linear time-varying differential equation. This last fact will be important for the later discussion. For any trajectory $p(t)$ that lies in N, columns of the matrix $D(p)$ form a dichotomic basis if the following two properties are satisfied:

Property 1:

For each $p \in N$, $\Lambda(p)$ has the block-triangular form

$$\Lambda(p) = \begin{bmatrix} \Lambda_s(p) & \Lambda_{\text{su}}(p) \\ 0 & \Lambda_u(p) \end{bmatrix} \tag{10}$$

where $\Lambda_s(p) \in \mathbb{R}^{n \times n}$, $\Lambda_u(p) \in \mathbb{R}^{n \times n}$ and $\Lambda_{\text{su}}(p) \in \mathbb{R}^{n \times n}$.

Property 2:

Along any segment $p(t)$, $t \in [t_1, t_2]$ of a trajectory $p(t)$, of $G(p)$ lying in N, the transition matrices $\Phi_s^p(t, 0)$ and $\Phi_u^p(t, 0)$ corresponding to Λ_s and Λ_u, respectively, defined such that $\Phi_s^p(0, 0) = I$ and $\Phi_u^p(0, 0) = I$, satisfy the inequalities

$$\left\| \Phi_s^p(t, 0) \left[\Phi_s^p(\sigma, 0) \right]^{-1} \right\| \le K_1 \| D[p(\sigma)] \| \| D^{-1}[p(t)] \| e^{-\alpha(t-\sigma)}$$

$$t_2 \ge t \ge \sigma > t_1$$

$$\left\| \Phi_u^p(t, 0) \left[\Phi_u^p(\sigma, 0) \right]^{-1} \right\| \le K_1 \| D[p(\sigma)] \| \| D^{-1}[p(t)] \| e^{-\alpha(\sigma-t)}$$

$$t_1 \le t \le \sigma < t_2 \tag{11}$$

where $K_1 > 0$ and $\alpha > 0$ are positive scalars that can vary along N (see Ref. 1). Properties 1 and 2 ensure that Φ_s^p contracts vectors exponentially in forward time, whereas Φ_u^p contracts vectors exponentially in backward time. Because the boundary layers of the three-segment approximation of Eq. (5) are finite, it is important that the exponential bounds be tight at $t = \sigma$. See Refs. 1 and 2 for further details.

Riccati Dichotomic Basis in Initial Boundary Layer

Let $t = t_{\text{ibl}}$ be large enough so that the three-segment approximation $\hat{p}(t)$ lies within a prespecified tolerance ϵ [as defined in Eq. (6)] of the optimal trajectory $p^*(t)$ of the HBVP of Eq. (4). Furthermore, let $p_s(t) = [x_s(t), \lambda_s(t)]$ be the segment of $\hat{p}(t)$ in the initial boundary layer. Consequently, $p_s(t)$ lies in the stable manifold of $\bar{p}$ and is a solution to the following HBVP:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} H_\lambda^T \\ -H_x^T \end{bmatrix}, \qquad \begin{bmatrix} x(0) = x_0 \\ x(t_{\text{ibl}}) = \bar{x} \end{bmatrix} \tag{12}$$

Let the Hamiltonian vector field of Eq. (12) be decomposed as

$$\begin{bmatrix} H_\lambda^T \\ -H_x^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} h_s \\ h_u \end{bmatrix} \tag{13}$$

where $P \equiv P[p_s(t)] \in \mathbb{R}^{n \times n}$. Differentiating along $p_s(t)$, we have

$$\begin{bmatrix} H_{\lambda x} & H_{\lambda\lambda} \\ -H_{xx} & -H_{x\lambda} \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} h_s \\ h_u \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \dot{P} & 0 \end{bmatrix} \begin{bmatrix} h_s \\ h_u \end{bmatrix} + \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} \dot{h}_s \\ \dot{h}_u \end{bmatrix} \tag{14}$$

Combining like terms, we obtain

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} \dot{h}_s \\ \dot{h}_u \end{bmatrix} = \begin{bmatrix} H_{\lambda x} + H_{\lambda\lambda} P & H_{\lambda\lambda} \\ -\dot{P} - H_{xx} - H_{x\lambda} P & -H_{x\lambda} \end{bmatrix} \begin{bmatrix} h_s \\ h_u \end{bmatrix} \tag{15}$$

Noting that

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}$$

we obtain

$$\begin{bmatrix} \dot{h}_s \\ \dot{h}_u \end{bmatrix} = \begin{bmatrix} H_{\lambda x} + H_{\lambda\lambda} P & H_{\lambda\lambda} \\ -\dot{P} - P H_{\lambda x} - H_{x\lambda} P - P H_{\lambda\lambda} P - H_{xx} & -H_{x\lambda} - P H_{\lambda\lambda} \end{bmatrix} \times \begin{bmatrix} h_s \\ h_u \end{bmatrix} \tag{16}$$

It can be seen from Eq. (16) that h_u is decoupled from h_s when P satisfies the following Riccati differential equation:

$$\dot{P} = -P H_{\lambda x} - H_{x\lambda} P - P H_{\lambda\lambda} P - H_{xx} \tag{17}$$

Equation (16) then simplifies to

$$\begin{bmatrix} \dot{h}_s \\ \dot{h}_u \end{bmatrix} = \begin{bmatrix} H_{\lambda x} + H_{\lambda\lambda} P & H_{\lambda\lambda} \\ 0 & -H_{x\lambda} - P H_{\lambda\lambda} \end{bmatrix} \begin{bmatrix} h_s \\ h_u \end{bmatrix} \tag{18}$$

Consequently, if Eq. (17) is satisfied it follows that property 1 of the preceding section is satisfied along $p_s(t)$.

Now let $\Phi_s(t, 0)$ and $\Phi_u(t, 0)$ be the transition matrices of $\Lambda_s \equiv H_{\lambda x} + H_{\lambda\lambda} P$ and $\Lambda_u \equiv -H_{x\lambda} - P H_{\lambda\lambda}$, respectively, along $p_s(t)$. The matrices $\Phi_s(t, 0)$ and $\Phi_u(t, 0)$ satisfy the properties

$$\Phi_s(t, 0)[\Phi_s(\sigma, 0)]^{-1} = \Phi_s(t, \sigma)$$
$$\Phi_u(t, 0)[\Phi_u(\sigma, 0)]^{-1} = \Phi_u(t, \sigma) \tag{19}$$

From the structure of Eq. (18), it can be seen that $\Lambda_u = -\Lambda_s^T$. Furthermore, because $p_s(t)$ lies in the stable manifold of $\bar{p}$ we have $P[p_s(t_{\text{ibl}})] \equiv \bar{P}$, where $\bar{P}$ satisfies the following Riccati algebraic equation:

$$-\bar{P} H_{\bar{\lambda}\bar{x}} - H_{\bar{x}\bar{\lambda}} P - P H_{\bar{\lambda}\bar{\lambda}} P - H_{\bar{x}\bar{x}} = 0 \tag{20}$$

The restriction on $p_s(t)$ together with the fact that P satisfies Eq. (17) with the boundary condition of Eq. (20) lead to the fact that $\Phi_s(t, 0)$ contracts vectors in forward time while $\Phi_u(t, 0)$ expands vectors in forward time. Consequently, along $p_s(t)$ there exist scalars K and α such that property 2 holds. Therefore, when P satisfies Eq. (17) together with the boundary condition of Eq. (20) the columns of the matrix

$$\begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \tag{21}$$

form a dichotomic basis along the segment $p_s(t)$ defined in Eq. (5).

Riccati Dichotomic Basis Method

Suppose for now that the solution of the Riccati differential equation $P(t)$ along the solution of Eq. (12) is known. Because $p_s(t)$ lies in the stable manifold of $(\bar{x},\bar{\lambda})$, the following condition must hold along $p_s(t)$:

$$h_u \equiv 0, \qquad 0 \le t \le t_{\text{ibl}} \tag{22}$$

In particular, at $t = 0$ we must have

$$h_u(0) = 0 \tag{23}$$

Using Eq. (13), the conditions of Eq. (23) are equivalent to the system of algebraic equations:

$$\begin{bmatrix} -P(0) & I \end{bmatrix} \begin{bmatrix} H_\lambda \\ -H_x \end{bmatrix} = 0 \tag{24}$$

Equation (24) is a system of n algebraic equations and can be solved for the initial adjoint $\lambda(0)$. Once $\lambda(0)$ has been determined, Eq. (13) can be used to solve for $h_s(0)$ via

$$h_s(0) = h_{s0} = H_\lambda(x_0, \lambda_0) \tag{25}$$

Setting $\lambda(0) \equiv \lambda_0$, the solution in the initial boundary layer can then be found by integrating the following system of differential equations from $t = 0$ to $t = t_{\text{ibl}}$ using the initial condition (x_0, λ_0, h_{s0}):

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{h}_s \end{bmatrix} = \begin{bmatrix} I \\ P \\ H_{\lambda x} + H_{\lambda\lambda} P \end{bmatrix} h_s \tag{26}$$

However, P is not known a priori; the solution $p_s(t)$ must be computed by successive approximation. This successive approximation procedure is given in the following algorithm:

Algorithm 1: This is the solution in initial boundary layer.

Set a convergence level δ and a matching tolerance ε.

1) Choose t_{ibl}.

2) Choose an initial guess $P(t)$ for the solution of the Riccati differential equation of Eq. (17), and use $P(t)$ to form the basis of Eq. (21) for the HBVP of Eq. (12). Furthermore, choose $P(t)$ so that $P(t_{\text{ibl}}) = \bar{P}$, where $\bar{P}$ satisfies Eq. (20).

3) Generate the missing initial conditions $\lambda(0) = \lambda_0$ and $h_s(0) = h_{s0}$ using Eqs. (24) and (25), respectively.

4) Using $P(t)$ and the initial conditions (x_0, λ_0, h_{s0}), integrate the following system of equations from $t = 0$ to $t = t_f$:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{h}_s \end{bmatrix} = \begin{bmatrix} I \\ P \\ H_{\lambda x} + H_{\lambda\lambda} P \end{bmatrix} h_s \tag{27}$$

5) Using the terminal condition $(\bar{x}, \bar{\lambda}, \bar{P})$ and $h_s(t)$ from step 2, integrate the following system of differential equations from $t = t_f$ to $t = 0$:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix} h_s, \qquad \dot{P} = -P H_{\lambda x} - H_{x\lambda} P - P H_{\lambda\lambda} P - H_{xx} \tag{28}$$

6) Using $P(t)$ from step 3, repeat steps 1–3 until $\|[x(t_{\text{ibl}}), \lambda(t_{\text{ibl}})]^{(k)} - [x(t_{\text{ibl}}), \lambda(t_{\text{ibl}})]^{(k-1)}\| < \delta$, where k is the kth iterate of step 2. If $\|[x(t_{\text{ibl}}), \lambda(t_{\text{ibl}})]^{(k)} - (\bar{x}, \bar{\lambda})\| < \varepsilon$, then stop. Otherwise, start again at step 1 with a larger value of t_{ibl}.

Initialization

As with any successive approximation procedure, the convergence of algorithm 1 is predicated on a sufficiently good initial guess. To date, algorithm 1 has been tested on several examples. For these examples it has been found that the solution of the Riccati algebraic equation at the equilibrium point $\bar{P}$ has been a good initial guess for use in algorithm 1. However, it has not been shown mathematically that the algorithm will always converge with $\bar{P}$ as an initial guess. For problems where algorithm 1 does not converge with $\bar{P}$ as an initial guesses, other initialization techniques must be explored. Such an investigation is beyond the scope of this Note.

Application of Method to Performance Optimization of Supersonic Aircraft

The Riccati dichotomic basis method of algorithm 1 is now applied to a problem in performance optimization of supersonic aircraft. Consider an aircraft flying in the vertical plane over a flat Earth. The longitudinal equations of motion are given as[7]

$$\dot{E} = (V/mg)(T - D), \qquad \epsilon\dot{h} = V \sin\gamma$$
$$\epsilon\dot{\gamma} = (g/V)(n - \cos\gamma) \tag{29}$$

where m is the vehicle mass, g is the acceleration caused by gravity, E is the energy altitude, h is the vehicle altitude, γ is the flight-path angle, $V = \sqrt{[2g(E - h)]}$ is the speed, $T = T(V, h)$ is the thrust, $D = D(V, h)$ is the drag, n is the load factor (i.e., the vertical component of lift), and ϵ is an artificial parameter that identifies the timescale separation. A complete description of the aerodynamics and thrust models is given in Ref. 7.

Optimal Control Problem

It is desired to steer the vehicle from an initial state $[E(0)\ h(0)\ \gamma(0)]$ to a terminal state $[E(t_f)\ h(t_f)\ \gamma(t_f)]$ while minimizing the time given by the cost functional

$$J = \int_0^{t_f} \mathrm{d}t \tag{30}$$

The Hamiltonian is given as

$$H = 1 + \lambda_E \dot{E} + \lambda_h \dot{h} + \lambda_\gamma \dot{\gamma} \tag{31}$$

The corresponding adjoint equations are given as

$$\dot{\lambda}_E = -H_E^*, \qquad \epsilon\dot{\lambda}_h = -H_h^*, \qquad \epsilon\dot{\lambda}_\gamma = -H_\gamma^* \tag{32}$$

where H^* is the Hamiltonian evaluated at the optimal control n^* (see Ref. 7 for details). The resulting HBVP consists of the differential equations of Eq. (29) and Eq. (32) together with the boundary conditions

$$E(0) = E_0, \qquad E(t_f) = E_f, \qquad h(0) = h_0, \qquad h(t_f) = h_f$$
$$\gamma(0) = \gamma_0, \qquad \gamma(t_f) = \gamma_f \tag{33}$$

Reduction of Order in the Left Boundary Layer

For an initial condition where $V/a \approx 1$ (where a is the local speed of sound) and terminal conditions where $V/a > 1$, the HBVP of Eqs. (32) and Eq. (33) becomes hypersensitive,[1,2] and the optimal trajectory is composed of a fast initial boundary-layer segment, a slow middle segment, and a fast terminal boundary-layer segment. Denoting the fast timescale by $\tau = t/\epsilon$, the dynamics can be written in terms of τ as

$$E' = \epsilon(V/mg)(T - D), \qquad h' = V\sin\gamma$$
$$\gamma' = (g/V)(n^* - \cos\gamma), \qquad \lambda_E' = -\epsilon H_E^*$$
$$\lambda_h' = -H_h^*, \qquad \lambda_\gamma' = -H_\gamma^* \tag{34}$$

where $(\cdot)'$ denotes differentiation with respect to τ.

Completely Hypersensitive HBVP

The solutions in the initial and terminal boundary layers have the same structure except that the directions of time are opposite. Consequently, it is sufficient to focus on the solution in the initial boundary layer. The zeroth-order approximation to the optimal trajectory in the initial boundary layer can be obtained by setting $\epsilon \equiv 0$ (see Ref. 8). The reduced-order HBVP on the fast timescale then becomes completely hypersensitive and consists of the Hamiltonian differential equations

$$h' = V\sin\gamma, \qquad \gamma' = (g/V)(n^* - \cos\gamma)$$
$$\lambda_h' = -H_h^*, \qquad \lambda_\gamma' = -H_\gamma^* \tag{35}$$

together with the boundary conditions

$$h(\tau = 0) = h_0, \qquad h(\tau_{\mathrm{ibl}}) = \bar{h}$$
$$\gamma(\tau = 0) = \gamma_0, \qquad \gamma(\tau_{\mathrm{ibl}}) = \bar{\gamma} = 0 \tag{36}$$

where τ_{ibl} is the end of the initial boundary-layer segment. The values of $E = E(0) = \text{constant}$ and $\lambda_E = \lambda_E(0) = \text{constant}$, $\bar{h}$, and $\bar{\gamma}$ correspond to an equilibrium condition at the end of the initial boundary layer to zeroth order on the fast timescale and are found using the method of matched asymptotic expansions as described in Ref. 8.

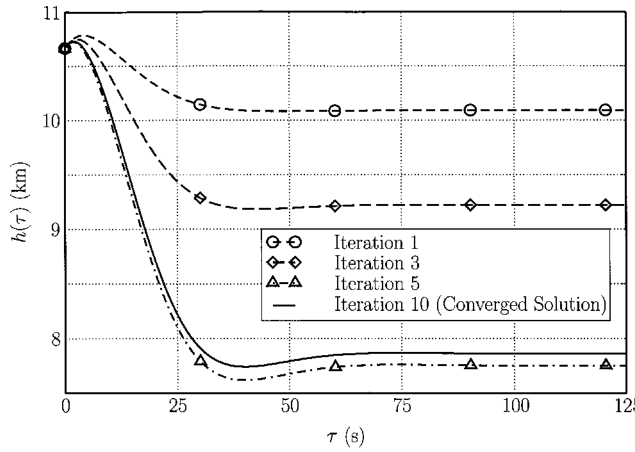

Fig. 1 Iterates of $h(\tau)$ vs τ in initial boundary layer.

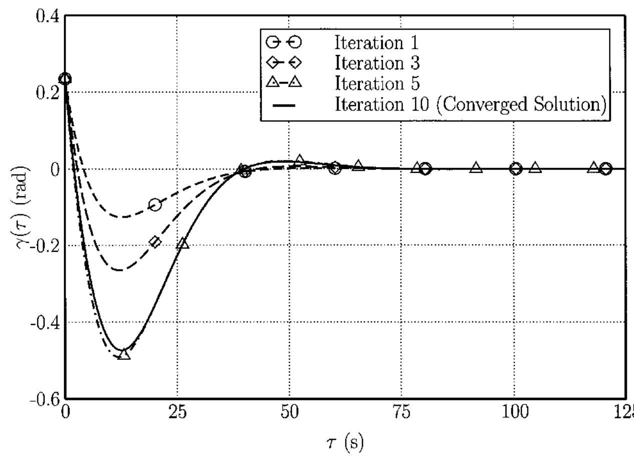

Fig. 2 Iterates of $\gamma(\tau)$ vs τ in initial boundary layer.

Application of Riccati Dichotomic Basis Method

Algorithm 1 is now applied to the HBVP of Eqs. (35) and (36). Typical values for E and λ_E are $E = 14{,}700$ m and $\lambda_E = -0.00667$, respectively. A typical initial condition for this problem is

$$h(\tau = 0) = 10{,}668\ \mathrm{m}, \qquad \gamma(\tau = 0) = 0.234\ \mathrm{rad} \tag{37}$$

The saddle point for this example is given as

$$\bar{h} = 7865\ \mathrm{m}, \qquad \bar{\gamma} = 0\ \mathrm{rad}, \qquad \bar{\lambda} = 0\ \mathrm{m}$$
$$\bar{\lambda}_\gamma = -1.335313969 \tag{38}$$

Finally, the parameters δ, ϵ, and τ_{ibl} (required for algorithm 1) are taken as 10^{-3}, 10^{-3}, and 125 s, respectively.

The results for h and γ are shown in Figs. 1 and 2, respectively (iteration 10 is the converged solution). Similar results are obtained for λ_h and λ_γ, but are not shown. In addition to convergence, one key feature of the method is illustrated by the numerical results. It is seen that each of the solution iterates levels off as $\tau \to \tau_{\mathrm{ibl}}$. Consequently, on the time interval $\tau \in [0, \tau_{\mathrm{ibl}}]$ has been eliminated. The success of the method on a relatively complex single timescale problem suggests that the dichotomic basis method might be extendable to more complex problems, including problems that evolve on multiple timescales.

Discussion of Method

One needs some a priori knowledge that an optimal control problem is completely hypersensitive before the dichotomic basis method should be applied. This knowledge can come from a solution obtained by a direct method. If the solution has the characteristic takeoff, cruise, landing structure, then the dichotomic basis method is applicable. Also, the particular saddle point that is influencing the nature of the solution can be identified. Completely hypersensitive HBVPs can involve multiple saddle points and (hetero-clinic) orbits that connect them. The solution from a direct method can be used to identify such structure, and the approach considered in this Note can be adapted to handle it.

A motivating reason for using the Riccati dichotomic basis method is to gain insight into the phase space structure in the neighborhood of the optimal solution. Previously, an approximate dichotomic basis method was developed to solve completely hypersensitive HBVPs[2] and has been shown to work successfully on problems of moderate complexity.[3] The major disadvantage of the approximate dichotomic basis method is that it does not produce direct information about the exact directions of expanding and contracting behavior in the tangent space of the optimal trajectory. This deficiency is overcome by the Riccati dichotomic basis method described here. Not only does the Riccati dichotomic basis method produce a solution to the HBVP, but it also produces the solution to the Riccati differential equations from which the dichotomic basis can be constructed with no further computation. The dichotomic basis provides useful information about the conditions satisfied by points on the stable and unstable manifolds.

Finally, it is important to distinguish the successive approximation procedures developed here from the well-known backward sweep method presented in Ref. 9, which also solves a Riccati differential equation. The basic difference between the two methods is that the method of Ref. 9 makes no attempt to eliminate the hypersensitivity, whereas the key feature of the current method is that the hypersensitivity is eliminated over the time interval of interest. Consequently, for problems with relatively long durations in the boundary layers the method of Ref. 9 will be less likely to succeed than will the method developed in this Note.

Conclusions

A Riccati dichotomic basis has been constructed in the initial boundary-layer segments along the solution of completely hypersensitive HBVPs arising in optimal control. The structure of this basis has led to the development of a successive approximation procedure to find the solution in the initial boundary layer. The successive approximation procedure was illustrated on a problem in supersonic aircraft flight, and its range of applicability was briefly discussed.

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Stability and Convergence of a Hybrid Adaptive Feedforward Observer

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Introduction

IN the active noise and vibration control field, as well as the field of flexible structures, it is common to require an estimation technique for predicting the appropriate system response where it is either physically impossible or undesirable to place an actual error sensor.[1–4] To resolve this virtual sensing estimation problem, especially for systems subjected to nonstationary tonal disturbances, Tran and Southward[5] proposed using a hybrid adaptive feedforward observer. This technique is a dynamic closed-loop transformation that estimates the system states using the sensor responses and a dynamic system model. Similar to other observer designs, the separation principle between the controller and the hybrid observer is confirmed.

The hybrid adaptive feedforward observer consists of two components. A conventional feedback component is used to stabilize the closed-loop observer, as well as to speed up the convergence process. In addition, it provides extra design freedom to minimize the effects of process and sensor noise. An adaptive feedforward component is used to track the unknown mapping of the nonstationary tonal disturbance onto the observer states. The adaptation was achieved using a least-mean-squares (LMS)-based gradient descent method.

In this Note, the stability and convergence of the hybrid observer is analyzed for the first time. It was proven that, in this design, the nonautonomous overall system dynamics indeed contains linear-time-invariant (LTI) eigenvalues. The analytical result on stability was demonstrated using a one-dimensional acoustic duct.

Problem Formulation

Consider a plant or dynamic system with a vector of outputs $\boldsymbol{y}$ obtained as the measured response from sensors, as shown in Fig. 1. A vector of unknown external disturbances $\boldsymbol{w}$ excites the plant as well as a vector of control inputs $\boldsymbol{u}$. No a priori knowledge of how the disturbance $\boldsymbol{w}$ actually affects the internal states of the plant is assumed; however, a feedforward reference signal, which is correlated to the disturbance in $\boldsymbol{w}$, is assumed to be available as indicated by the dotted line in the left half of Fig. 1.

The plant is represented in state-space notation with the mapping of the unknown external disturbance explicitly shown in the state equation as

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{w}, \qquad \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}\boldsymbol{u} \tag{1}$$

Provided that the state-space model $(\boldsymbol{A}, \boldsymbol{C})$ is completely observable,[6] we first design a conventional observer to estimate the system states, assuming no disturbance input, that is, $\boldsymbol{w} = \boldsymbol{0}$. The conventional observer is then augmented with an adaptive feedforward

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